Nash equilibria in Voronoi games on graphs

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Abstract

In this paper we study a game where every player is to choose a vertex (facility) in a given undirected graph. All vertices (customers) are then assigned to closest facilities and a player's payoff is the number of customers assigned to it. We show that deciding the existence of a Nash equilibrium for a given graph is $\mathcal{NP}$-hard which to our knowledge is the first result of this kind for a zero-sum game. We also introduce a new measure, the *social cost discrepancy*, defined as the ratio of the costs between the worst and the best Nash equilibria. We show that the social cost discrepancy in our game is $\Omega(\sqrt{n/k})$ and $O(\sqrt{kn})$, where n is the number of vertices and k the number of players.

1 Introduction

Voronoi game is a widely studied game which plays on a continuous space, typically a 2-dimensional rectangle. Players alternatively place points in the space. Then the Voronoi diagram is considered. Every player gains the total surface of the Voronoi cells of his points [1]. This game is related to the facility location problem, where the goal is to choose a set of k facilities in a bipartite graph, so to minimize the sum of serving cost and facility opening cost [8].

We consider the discrete version of the Voronoi game which plays on a given graph instead on a continuous space. Whereas most papers about these games [3, 5] study the existence of a winning strategy, or computing the best strategy for a player, we study in this paper the Nash equilibria.

Formally the discrete Voronoi game plays on a given undirected graph $G(V, E)$ with $n = |V|$ and k players. Every player has to choose a vertex (*facility*) from V, and every vertex (*customer*) is assigned to the closest facilities. A player's payoff is the number of vertices assigned to his facility. We define the *social cost* as the sum of the distances to the closest facility over all vertices.

We consider a few typical questions about Nash Equilibria:

- Do Nash equilibria exist?
- What is the computational complexity for finding one?
- If they exist, can one be found from an arbitrary intital strategy profile with the best-response dynamic?
- It they exist, how different are their social costs?

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The existence of Nash equilibria is a graph property for a fixed number of players, and we give examples of graphs for which there exist Nash equilibria and examples for which there are none. We show that deciding this graph property is an $\mathcal{NP}$-hard problem.

For particular graphs, namely the cycles, we characterize all Nash equilibria. We show that the best-response dynamic does not converge to a Nash equilibria on these graphs, but does for a modified version of this game.

Finally we introduce a new measure. Assume that there are Nash equilibria for k players on graph G. Let A be the largest social cost of a Nash equilibrium, B the smallest social cost of a Nash equilibrium, and C the smallest social cost of any strategy profile, which is not necessarily an equilibrium. Then the ratio A/C is called the price of anarchy [7], B/C is called the price of stability [2]. We study a different ratio A/B, which we call the *social cost discrepancy* of G. The social cost discrepancy of the game is defined as the worst discrepancy over all instances of the game. The idea is that a small social cost discrepancy guarantees that the social costs of Nash equilibria do not differ too much, and measures a degree of choice in the game. In some settings it may be unfair to compare the cost of a Nash equilibrium with the optimal cost, which may not be attained by selfish agents. Note that this ratio is upper-bounded by the price of anarchy. We show that the social cost discrepancy in our game is $\Omega(\sqrt{n/k})$ and $O(\sqrt{kn})$. Hence for a constant number of players we have tight bounds.

2 The game

For this game we need to generalize the notion of vertex partition of a graph: A *generalized partition* of a graph $G(V,E)$ is a set of n-dimensional non-negative vectors, which sum up to the vector with 1 in every component, for $n = |V|$.

The Voronoi game on graphs consists of:

- A graph $G(V,E)$ and k players. We assume $k < n$ for $n = |V|$, otherwise the game has a trivial structure. The graph induces a distance between vertices $d : V \times V \to \mathbb{N} \cup \{\infty\}$, which is defined as the minimal number of edges of any connecting path, or infinite if the vertices are disconnected.

- The strategy set of each player is V. A strategy profile of k players is a vector $f = (f_1, \ldots, f_k)$ associating each player to a vertex.

- For every vertex $v \in V$ — called *customer* — the distance to the closest facility is denoted as $d(v,f) := \min_{f_i} d(v, f_i)$. Customers are assigned in equal fractions to the closest facilities as follows. The strategy profile f defines the generalized partition $\{F_1, \ldots, F_k\}$, where for every player $1 \le i \le k$ and every vertex $v \in V$,

$$F_{i,v} = \begin{cases} \frac{1}{|\arg\min_j d(v,f_j)|} & \text{if } d(v,f_i) = d(v,f) \\ 0 & \text{otherwise.} \end{cases}$$

  We call F_i the *Voronoi cell* of player i. Now the payoff of player i is the (fractional) amount of customers assigned to it (see figure 1), that is $p_i := \sum_{v \in V} F_{i,v}$.

The *best response* for player i in the strategy profile f is a vertex $f_i' \in V$ maximizing the player i's payoff in the strategy profile (f_{-i}, f_i') which is a shorthand for the profile which equals f, except

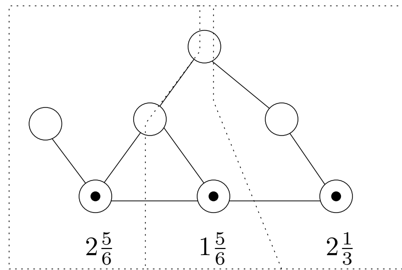

Figure 1: A strategy profile of a graph (players are dots) and the corresponding payoffs.

that the strategy of player i is f'_i. If player i can strictly improve his payoff by choosing another strategy, we say that player i is *unhappy* in f, otherwise he is *happy*. The *best response dynamic* is the process of repeatedly choosing an arbitrary unhappy player, and change it to an arbitrary best response. A *pure Nash equilibrium* is defined as a fixed point to the best response dynamic, or equivalently as a strategy profile where all players are happy. In this paper we consider only pure Nash equilibria, so we omit from now on the adjective "pure".

We defined players' payoffs in such a way, that there is a subtle difference between the Voronoi game played on graphs and the Voronoi game played on a continuous surface. Consider a situation where a player i moves to a location already occupied by a single player j, then in the continuous case player i gains exactly a half of the previous payoff of player j (since it is now shared with i). However, in our setting (the discrete case), player i can sometimes gain more than a half of the previous payoff of player j (see figure 2).

Also note that the best responses for a player in our game are computable in polynomial time, whereas for the Voronoi game in continuous space, the problem seems hard [4].

A simple observation leads to the following bound on the players payoff.

Lemma 1 *In a Nash equilibrium the payoff p_i of every player i is bounded by $n/2k < p_i < 2n/k$.*

Proof: If a player gains p and some other player moves to the same location then both payoffs are at least $p/2$. Therefore the ratio between the largest and the smallest payoffs among all players can be at most 2. If all players have the same payoff, it must be exactly n/k, since the payoffs sum up to n. Otherwise there is at least one player who gains strictly less than n/k, and another player who gains strictly more than n/k. This concludes the proof. □

3 Example: the cycle graph

Let $G(V, E)$ be the cycle on n vertices with $V = \{v_i : i \in \mathbb{Z}_n\}$ and $E = \{(v_i, v_{i+1}) : i \in \mathbb{Z}_n\}$, where addition is modulo n. The game plays on the undirected cycle, but it will be convenient to fix an orientation. Let $u_0, \ldots, u_{\ell-1}$ be the distinct facilities chosen by k players in a strategy profile f with $\ell \leq k$, numbered according to the orientation of the cycle. For every $j \in \mathbb{Z}_\ell$, let $c_j \geq 1$ be the number of players who choose the facility u_j and let $d_j \geq 1$ be the length of the directed path from u_j to u_{j+1} following the orientation of G. Now the strategy profile is defined by these 2ℓ numbers, up to permutation of the players. We decompose the distance into $d_j = 1 + 2a_j + b_j$, for $0 \leq b_j \leq 1$, where $2a_j + b_j$ is the number of vertices between facilities u_j and u_{j+1}. So if $b_j = 1$, then there is a vertex in midway at equal distance from u_j and u_{j+1}.

With these notations the payoff of player i located on facility u_j is

$$p_i := \frac{b_{j-1}}{c_{j-1}+c_j} + \frac{a_{j-1}+1+a_j}{c_j} + \frac{b_j}{c_j+c_{j+1}}.$$

All Nash equilibria are explicitly characterized by the following lemma. The intuition is that the cycle is divided by the players into segments of different length, which roughly differ at most by a factor 2. The exact statement is more subtle because several players can be located at a same facility and the payoff is computed differently depending on the parity of the distances between facilities.

Lemma 2 *For a given strategy profile, let γ be the minimal payoff among all players, i.e: $\gamma := \min\{p_i | 1 \le i \le k\}$. Then this strategy profile is a Nash equilibrium if and only if, for all $j \in \mathbb{Z}_\ell$:*

(i) $c_j \le 2$

(ii) $d_j \le 2\gamma$

(iii) If $c_j = 1$ and $d_{j-1} = d_j = 2\gamma$ then $c_{j-1} = c_{j+1} = 2$.

(iv) If $c_{j-1} = 2, c_j = 1, c_{j+1} = 1$ then d_{j-1} is odd.
If $c_{j-1} = 1, c_j = 1, c_{j+1} = 2$ then d_j is odd.

Lemma 3 *On the cycle graph, the best response dynamic does not converge.*

Proof: Figure 2 shows an example of a graph, where the best response dynamic can iterate forever. □

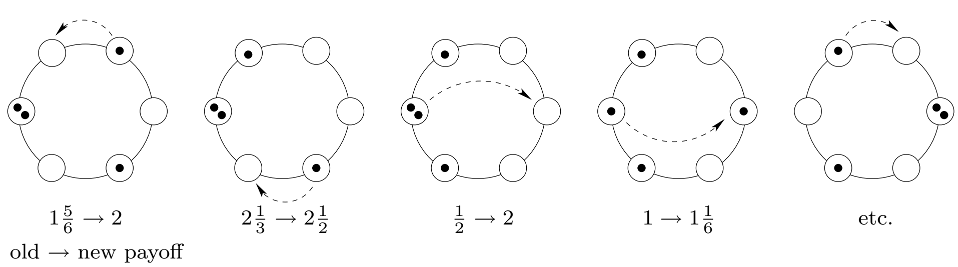

Figure 2: The best response dynamic does not converge on this graph.

However there is a slightly different Voronoi game in which the best response dynamic converges : The *Voronoi game with disjoint facilities* is identical with the previous game, except that players who are located on the same facility now gain zero.

Lemma 4 *On the cycle graph, for the* Voronoi game with disjoint facilities*, the best response dynamic does converge on a strategy profile in which players are located on distinct facilities.*

Proof: To show convergence we use a potential function. For this purpose we define the *dominance order*: Let A, B be two multisets. If $|A| < |B|$ then $A \succ B$. If $|A| = |B| \ge 1$, and $\max A > \max B$ then $A \succ B$. If $|A| = |B| \ge 1, \max A = \max B$ and $A \backslash \{\max A\} \succ B \backslash \{\max B\}$ then $A \succ B$. This is a total order.

The potential function is the multiset $\{d_0, d_1, \ldots, d_{k-1}\}$, that is all distances between successive occupied facilities. Player i's payoff — renumbered conveniently — is simply $(d_i + d_{i+1})/2$. Now consider a best response for player i moving to a vertex not yet chosen by another player, say between player j and $j+1$. Therefore in the multiset $\{d_0, d_1, \ldots, d_{k-1}\}$, the values d_i, d_{i+1}, d_j are replaced by $d_i + d_{i+1}, d', d''$ for some values $d', d'' \geq 1$ such that $d_j = d' + d''$. The new potential value is dominated by the previous one. This proves that after a finite number of iterations, the best response dynamic converges to a Nash equilibrium. □

4 Existence of a Nash equilibrium is $\mathcal{NP}$-hard

In this section we show that it is $\mathcal{NP}$-hard to decide whether for a given graph $G(V, E)$ there is a Nash equilibrium for k players. For this purpose we define a more general but equivalent game, which simplifies the reduction.

In the *generalized Voronoi game* $\langle G(V, E), U, w, k\rangle$ we are given a graph G, a set of facilities $U \subseteq V$, a positive weight function w on vertices and a number of players k. Here the set of strategies of each player is only U instead of V. Also the payoff of a player is the weighted sum of fractions of customers assigned to it, i.e. the payoff of player i is $p_i := \sum_{v\in V} w(v)F_{i,v}$.

Lemma 5 *For every generalized Voronoi game $\langle G(V, E), U, w, k\rangle$ there is a standard Voronoi game $\langle G'(V', E'), k\rangle$ with $V \subseteq V'$, which has the same set of Nash equilibria and which is such that $|V'|$ is polynomial in $|V|$ and $\sum_{v\in V} w(v)$.*

Proof: To construct G' we will augment G in two steps. Start with $V' = V$.

First, for every vertex $u \in V$ such that $w(u) > 1$, let H_u be a set of $w(u) - 1$ new vertices. Set $V' = V' \cup H_u$ and connect u with every vertex from H_u.

Second, let H be a set of $k(a + 1)$ new vertices where $a = |V'| = \sum_{v\in V} w(v)$. Set $V' = V' \cup H$ and connect every vertex of U with every vertex of H.

Now in $G'(V', E')$ every player's payoff can be decomposed in the part gained from $V'\backslash H$ and the part gained from H. We claim that in a Nash equilibrium every player chooses a vertex from U. If there is at least one player located in U, then the gain from H of any other player is 0 if located in $V'\backslash(U \cup H)$, is 1 if located in H and is at least $a + 1$ if located in U. Since the total payoff from $V'\backslash H$ over all players is a, this forces all players to be located in U.

Clearly by construction, for any strategy profile $f \in U^k$, the payoffs are the same for the generalized Voronoi game in G as for the standard Voronoi game in G'. Therefore we have equivalence of the set of Nash equilibria in both games. □

Our $\mathcal{NP}$-hardness proof will need the following gadget.

Lemma 6 *For the graph G shown in figure 3 and $k = 2$ players, there is no Nash equilibrium.*

Proof: We will simply show that given an arbitrary location of one player, the other player can move to a location where he gains at least 5. Since the total payoff over both players is 9, this will prove that there is no Nash equilibrium, since the best response dynamic does not converge.

By symmetry without loss of generality the first player is located at the vertices u_1 or u_2. Now if the second player is located at u_6, his payoff is at least 5. □

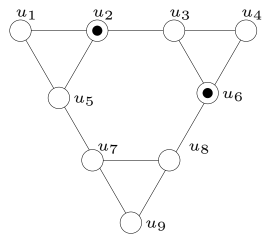

Figure 3: Example of a graph with no Nash equilibrium for 2 players.

Theorem 1 *Given a graph $G(V, E)$ and a set of k players, deciding the existence of Nash equilibrium for k players on G is $\mathcal{NP}$-complete.*

Proof: The problem is clearly in $\mathcal{NP}$, since best responses can be computed in polynomial time, therefore it can be verified efficiently if a strategy profile is a Nash equilibrium.

The proof of $\mathcal{NP}$-hardness is by the reduction from 3-PARTITION, which is unary $\mathcal{NP}$-complete [6]. In this later problem we are given integers $a_1, \ldots, a_{3m}$ and B such that $B/4 < a_i < B/2$ for every $1 \le i \le 3m$, $\sum_{i=1}^{3m} = mB$ and have to partition them into disjoint sets $P_1, \ldots, P_m \subseteq \{1, \ldots, 3m\}$ such that for every $1 \le j \le m$ we have $\sum_{i \in P_j} a_i = B$.

We construct a weighted graph $G(V, E)$ with the weight function $w : V \to \mathbb{N}$ and a set $U \subseteq V$ such that for $k = m + 1$ players ($m \ge 2$) there is a Nash equilibrium to the generalized Voronoi game $\langle G, U, w, k\rangle$ if and only if there is a solution to the 3-PARTITION instance. We define the constants $c = \binom{3m}{3} + 1$ and $d = \left\lfloor \frac{Bc - c + c/m}{5} \right\rfloor + 1$. The graph G consists of 3 parts. In the first part V_1, there is for every $1 \le i \le 3m$ a vertex v_i of weight $a_i c$. There is also an additional vertex v_0 of weight 1. In the second part V_2, there is for every triplet (i, j, k) with $1 \le i < j < k \le 3m$ a vertex u_{ijk} of unit weight. — Ideally we would like to give it weight zero, but there seems to be no simple generalization of the game which allows zero weights, while preserving the set of Nash equilibria. — Every vertex u_{ijk} is connected to v_0, v_i, v_j and v_k. The third part V_3, consists of the 9 vertex graph of figure 3 where each of the vertices $u_1, \ldots, u_9$ has weight d. To complete our construction, we define the facility set $U := V_2 \cup V_3$.

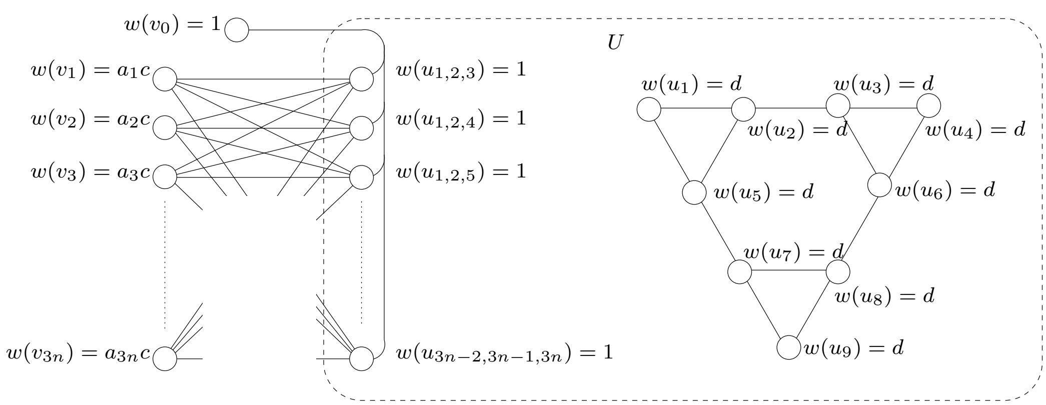

Figure 4: Reduction from 3-PARTITION.

First we show that if there is a solution $P_1, \ldots, P_m$ to the 3-PARTITION instance then there is a Nash equilibrium for this graph. Simply for every $1 \le q \le m$ if $P_q = \{i, j, k\}$ then player q is

assigned to the vertex u_{ijk}. Player $m+1$ is assigned to u_2. Now player $(m+1)$'s payoff is $9d$, and the payoff of each other player q is $Bc + c/m$. To show that this is a Nash equilibrium we need to show that no player can increase his payoff. There are different cases. If player $m+1$ moves to a vertex u_{ijk}, his payoff will be at most $\frac{3}{4}Bc + c/(m+1) < 9d$, no matter if that vertex was already chosen by another player or not. If player $1 \leq q \leq 3m$ moves from vertex u_{ijk} to a vertex u_ℓ then his gain can be at most $5d < Bc + c/m$. But what can be his gain, if he moves to another vertex $u_{i'j'k'}$? In case where $i = i', j = j', k \neq k'$, $a_i c + a_j c$ is smaller than $\frac{3}{4}Bc$ because $a_i + a_j + a_k = B$ and $a_k > B/4$. Since $a_{k'} < B/2$, and player q gains only half of it, his payoff is at most $a_i c + a_j c + a_{k'} c/2 + c/m < Bc + c/m$ so he again cannot improve his payoff. The other cases are similar.

Now we show that if there is a Nash equilibrium, then it corresponds to a solution of the 3-Partition instance. So let there be a Nash equilibrium. First we claim that there is exactly one player in V_3. Clearly if there are 2 players, this contradicts equilibrium by Lemma 6. If there are 3 players or more, then by a counting argument there are vertices v_i, v_j, v_k which are at distance more than one from any player. One of the players located at V_3 gains at most $3d$ and if he moves to u_{ijk}, his payoff would be at least $\frac{3}{4}Bc + c/m > 3d$. Now if there is no player in V_3, then any player moving to u_2 will gain $9d > \frac{3}{2}Bc + c/m$ which is an upper bound for the payoff of players located in V_2. So we know that there is a single player in V_3 and the m players in V_2 must form a partition, since otherwise there is a vertex $v_i \in V_1$ at distance at least 2 to any player. So, by the previous argument, there would be a player in V_2 who can increase his payoff by moving to the other vertex in V_2 as well. (He moves in such a way that his new facility is at distance 1 to v_i.) Moreover, in this partition, each player gains exactly $Bc + c/m$ because if one gains less, given all weights in V_1 are multiple of c, he gains at most $Bc - c + c/m$ and he can always augment his payoff by moving to V_3 ($5d > Bc - c + c/m$). □

5 Social cost discrepancy

In this section, we study how much the social cost of Nash equilibria can differ for a given graph, assuming Nash equilibria exist. We define the *social cost* of a strategy profile f as $\text{cost}(f) := \sum_{v \in V} d(v, f)$. Since we assumed $k < n$ the cost is always non-zero. The *social cost discrepancy* of the game is the maximal fraction $\text{cost}(f)/\text{cost}(f')$ over all Nash equilibria f, f'. For unconnected graphs, the social cost can be infinite, and so can be the social cost discrepancy. Therefore in this section we consider only connected graphs.

Lemma 7 *There are connected graphs for which the social cost discrepancy is $\Omega(\sqrt{n/k})$, where n is the number of vertices and k the number of players.*

Proof: We construct a graph G as shown in figure 5. The total number of vertices in the graph is $n = k(2a + b + 2)$. We distinguish two strategy profiles f and f': the vertices occupied by f are marked with round dots, and the vertices of f' are marked with triangular dots.

By straightforward verification, it can be checked that both f and f' are Nash equilibria. However the social cost of f is $\Theta(kb + ka^2)$ while the social cost of f' is $\Theta(kab + ka^2)$. The ratio between both costs is $\Theta(a) = \Theta(\sqrt{n/k})$ when $b = a^2$ and thus the cost discrepancy is lower bounded by this quantity. □

The *radius* of the Voronoi cell of player i is defined as $\max_v d(v, f_i)$ where the maximum is taken over all vertices v such that $F_{i,v} > 0$. The *Delaunay triangulation* is a graph H_f on the k

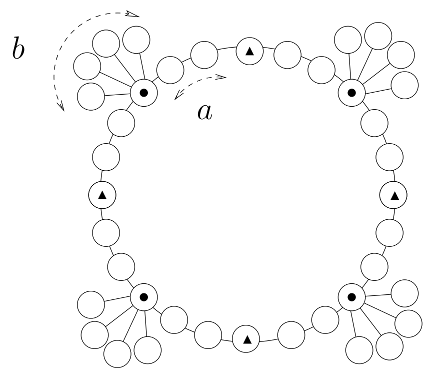

Figure 5: Example of a graph with high social cost discrepancy.

players. There is an edge (i, j) in H_f either if there is a vertex v in G with $F_{i,v} > 0$ and $F_{j,v} > 0$ or if there is an edge (v, v') in G with $F_{i,v} > 0$ and $F_{j,v'} > 0$.

We will need to partition the Delaunay triangulation into small sets, which are 1-dominated and contain more than one vertex. We call these sets *stars*: For a given graph $G(V, E)$ a vertex set $A \subseteq V$ is a *star* if $|A| \geq 2$, and there is a *center* vertex $v_0 \in A$ such that for every $v \in A, v \neq v_0$ we have $(v_0, v) \in E$. Note that our definition allows the existence of additional edges between vertices from A.

Lemma 8 *For any connected graph $G(V, E)$, V can be partitioned into stars.*

Proof: We define an algorithm to partition V into stars.

As long as the graph contains edges, we do the following. We start choosing an edge: If there is a vertex u with a unique neighbor v, then we choose the edge (u, v); otherwise we choose an arbitrary edge (u, v). Consider the vertex set consisting of u, v as well as of any vertex w that would be isolated when removing edge (u, v). Add this set to the partition, remove it as well as adjacent edges from G and continue.

Clearly the set produced in every iteration is a star. Also when removing this set from G, the resulting graph does not contain an isolated vertex. This property is an invariant of this algorithm, and proves that it ends with a partition of G into stars. □

Note that, when a graph is partitioned into stars, the centers of these stars form a dominating set of this graph. Nevertheless, vertices in a dominating set are not necessarily centers of any star-partition of a given graph.

The following lemma states that given two different Nash equilibria f and f', every player in f is not too far from some player in f'. For this purpose we partition the Delaunay triangulation H_f into stars, and bound the distance from any player of a star to f' by some value depending on the star.

Lemma 9 *Let f be an equilibrium and A be a star of a star partition of the Delaunay triangulation H_f. Let r be the maximal radius of the Voronoi cells over all players $i \in A$. Then, for any equilibrium f', there exists a player f'_j such that $d(f_i, f'_j) \leq 6r$ for every $i \in A$.*

Proof: Let $U = \{v \in V : \min_{i \in A} d(v, f_i) \leq 4r\}$. If we can show that there is a facility $f'_j \in U$ we would be done, since by definition of U there would be a player $i \in A$ such that $d(f_i, f'_j) \leq 4r$ and the distance between any pair of facilities of A is at most $2r$. This would conclude the lemma.

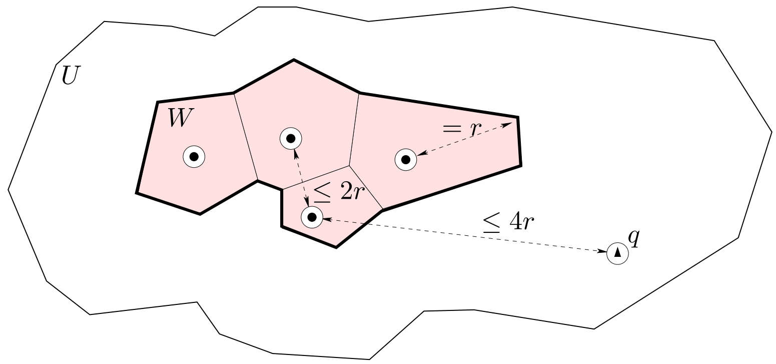

Figure 6: Illustration of lemma 9.

So for a proof by contradiction, assume that in the strategy profile f' there is no player located in U. Now consider the player with smallest payoff in f'. His payoff is not more than n/k. However if this player would choose as a facility the center of the star A, then he would gain strictly more: By the choice of r, any vertex in W is at distance at most $3r$ to the center of the star. However, by assumption and definition of U, any other facility of f' would be at distance strictly greater than $3r$ to any vertex in W. So the player would gain at least all vertices around it at distance at most $3r$, which includes W. Since any player's payoff is strictly more than $n/2k$ by Lemma 1, and since a star contains at least two facilities by definition, this player would gain strictly more than n/k, contradicting that f' is an equilibrium. This concludes the proof. □

Theorem 2 *For any connected graph $G(V, E)$ and any number of players k the social cost discrepancy is $O(\sqrt{kn})$, where $n = |V|$.*

Proof: Let f, f' be arbitrary equilibria on $G(V, E)$. We will consider a generalized partition of V and for each part bound the cost of f' by $c\sqrt{kn}$ times the cost of f for some constant c.

For a non-negative n-dimensional vector W we define the cost restricted to W as $\text{cost}_W(f) = \sum_{v\in V} W_v \cdot d(v, f)$. Now the cost of f would write as the sum of $\text{cost}_W(f)$ over the vectors W from some fixed generalized partition.

Fix a star partition of the Delaunay triangulation H_f. Let A be an arbitrary star from this partition, $a = |A|$, and W be the sum of the corresponding Voronoi cells, i.e. $W = \sum_{i\in A} F_i$. We will show that $\text{cost}_W(f') = O(\sqrt{kn} \cdot \text{cost}_W(f))$, which would conclude the proof. There will be two cases $k \leq n/4$ and $k > n/4$.

By the previous lemma there is a vertex f'_j such that $d(f_i, f'_j) \leq 6r$ for all $i \in A$, where r is the largest radius of all Voronoi cells corresponding to the star A. So the cost of f' restricted to the vector W is

$$\begin{aligned}
\text{cost}_W(f') &= \sum_{v\in V} W_v \cdot d(v, f') \leq \sum_{v\in V} W_v \cdot d(v, f'_j) \\
&= \sum_{v\in V}\sum_{i\in A} F_{i,v} \cdot d(v, f'_j) \\
&\leq \sum_{v\in V}\sum_{i\in A} F_{i,v} \cdot \big(d(v, f_i) + d(f_i, f'_j)\big) \\
&\leq \text{cost}_W(f) + 6r \cdot |W|, \qquad (1)
\end{aligned}$$

where $|W| := \sum_{v \in V} W_v$.

Moreover by definition of the radius, there is a vertex v with $W_v > 0$ such that the shortest path to the closest facility in A has length r. So the cost of f restricted to W is bigger than the cost restricted to this shortest path:

$$\text{cost}_W(f) \geq (\frac{1}{k} \cdot 1 + \frac{1}{k} \cdot 2 + \ldots + \frac{1}{k} \cdot r) \geq \frac{1}{k} \cdot r(r-1)/2.$$

(The fraction $\frac{1}{k}$ appears because a vertex can be assigned to at most k players.)

First we consider the case $k \leq n/4$. We have

$$\text{cost}_W(f) \geq |W| - |A| \geq a(n/2k - 1) \geq an/4k.$$

The first inequality is because the distance of all customers which are not facilites to a facility is at least one. The second inequality is due to Lemma 1 and $|W|$ is the sum of payoffs of all players in A.

Note that $|W| \leq n$ and $2 \leq a \leq k$. Now if $r \leq \sqrt{an}$, then

$$\frac{\text{cost}_W(f')}{\text{cost}_W(f)} \leq 1 + \frac{6r|W|}{\text{cost}_W(f)} \leq 1 + \frac{6r \cdot a \cdot 2n/k}{an/4k} = O(r) = O(\sqrt{kn}).$$

And if $r \geq \sqrt{an}$, then

$$\frac{\text{cost}_W(f')}{\text{cost}_W(f)} \leq 1 + \frac{6r|W|}{\text{cost}_W(f)}) \leq 1 + \frac{6r \cdot a \cdot 2n/k}{r(r-1)/2k} = O(an/r) = O(\sqrt{kn}).$$

Now we consider case $k > n/4$. In any equilibrium, the maximum payoff is at most $2n/k$. Moreover the radius r of any Voronoi cell is upper bounded by $n/k + 1$, otherwise the player with minimum gain (which is at most n/k) could increase his gain by moving to a vertex which is at distance at least r from every other facility. Therefore $r = O(1)$. Summing (1) over all stars with associated partition W, we obtain $\text{cost}(f') \leq \text{cost}(f) + cn$, for some constant c. Remark that the social cost of any equilibrium is at least $n - k$. Hence, $\frac{\text{cost}(f')}{\text{cost}(f)} = O(n)$. □

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Appendix

Lemma 2 *For a given strategy profile, let γ be the minimal payoff among all players, i.e: $\gamma := \min\{p_i | 1 \leq i \leq k\}$. Then this strategy profile is a Nash equilibrium if and only if, for all $j \in \mathbb{Z}_\ell$:*

(i) $c_j \leq 2$

(ii) $d_j \leq 2\gamma$

(iii) If $c_j = 1$ and $d_{j-1} = d_j = 2\gamma$ then $c_{j-1} = c_{j+1} = 2$.

(iv) If $c_{j-1} = 2, c_j = 1, c_{j+1} = 1$ then d_{j-1} is odd.
If $c_{j-1} = 1, c_j = 1, c_{j+1} = 2$ then d_j is odd.

Proof: (Necessary) We will show that if a strategy profile does not satisfy one of the conditions then it is not a Nash equilibrium.

(i) Suppose that there is a vertex u_j with $c_j \geq 3$. Assume $d_{j-1} \leq d_j$, the other case is symmetric. Since there are vertices on the cycle which are not occupied by a player, the payoff of each player located on u_j must be at least 1. Therefore $d_j \geq 3$. Let u' be the vertex immediately after u_j in the cycle. We show now that one of the players located at u_j can move to u' and strictly increase his payoff, which would contradict that the strategy profile is a Nash equilibrium. We decompose the distances into $d_{j-1} = 2a_{j-1} + b_{j-1} + 1$ and $d_j = 2a_j + b_j + 1$ where $0 \leq b_{j-1}, b_j \leq 1$. By $d_j \geq 3$ we have $a_j \geq 1$. By $d_{j-1} \leq d_j$ we have $a_{j-1} \leq a_j$. Now the payoff of a player in facility u_j is

$$\frac{b_{j-1}}{c_{j-1} + c_j} + \frac{a_{j-1} + 1 + a_j}{c_j} + \frac{b_j}{c_j + c_{j+1}}.$$

And if he moves to u', his payoff would be

$$a_j + b_j + \frac{1 - b_j}{c_j}.$$

In both cases $b_j = 0$ or $b_j = 1$ this new payoff is strictly greater.

(ii) Suppose that there exists j such that $d_j \geq 2\gamma+1$. As previous, we decompose $d_j = 2a+b+1$ where $0 \leq b \leq 1$. Assume that $c_j \leq c_{j+1}$, the other case is symmetric. Let u' be a vertex between u_j and u_{j+1} and at even distance to u_{j+1}. Note that the number of vertices between u_j and u_{j+1} except u' is $2a+b-1 = 2(a+b-1)+(1-b)$. If a player moves to u' he will gain at least $a+b+\frac{1-b}{1+c_j}$. Since $2a+b+1 \geq 2\gamma+1$, this payoff always greater than γ, so the player whose payoff is γ has an incentive to dislocate, contradicting that the strategy profile is a Nash equilibrium.

(iii) Assume that $c_j = 1$, $d_{j-1} = d_j = 2\gamma$ and at least one of c_{j-1}, c_j is 1. If a player whose payoff is γ moves to facility u_j, he will gain at least $\gamma - \frac{1}{2} + \frac{1}{c_{j-1}+2} + \frac{1}{c_j+2}$, which is strictly larger than γ.

(iv) Assume $c_j = 1$ and $c_{j-1} = 2, c_{j+1} = 1$, the other case is symmetric. Now if the distance d_{j-1} is even, the player located at u_j can stricly increase his payoff by moving to a vertex between u_{j-1} and u_j and at odd distance to u_{j-1}. The idea is that if this player gets a fractional payoff from some mid-way vertex, he'll better off sharing it with one, rather than two players.

(Sufficient) We will prove that if a strategy profile satisfies all conditions, then it is a Nash equilibrium.

(i) By (iii), if a player is alone on his facility then he has no incentive to make a *local move* — a move such that the set of his neighbors doesn't change.

(ii) Without loss of generality, assume that $c_j \leq c_{j+1}$. If a player moves to a vertex between facilities u_j and u_{j+1}, similarly as above, his payoff will be at most $1 + (a+b-1) + \frac{1-b}{c_j+1}$ where $d_j = 2a+b+1$. Since $d_j \leq 2\gamma$, this new payoff will be at most γ so the player has no incentive to move.

(iii) If a player moves from facility u_i to facility u_j which is not his neighbor (i.e, $i \notin \{j-1, j\}$), he will gains $\frac{b_{j-1}}{c_{j-1}+c_j+1} + \frac{a_{j-1}+1+a_j}{c_j+1} + \frac{b_j}{c_j+1+c_{j+1}}$ where $d_{j-1} = 2a_{j-1}+b_{j-1}+1$ and $d_j = 2a_j+b_j+1$. Again, since (iii), this payoff is less or equal γ.

(iv) Suppose that a player j (from facility u_j) moves to one of his facility-neighbor u_{j+1}. If $c_j > 1$ then the situation is as in the above paragraph. If $c_j = 1$, the distance between facilities u_{j-1} and u_{j+1} becomes $d_{j-1}+d_j$ and number of vertices between these two facilities can be decomposed as $2(a_{j-1}+a_j)+1$ if $b_{j-1} = b_j = 0$ or as $2(a_{j-1}+a_j+1)+(b_{j-1}+b_j-1)$ otherwise. Suppose that this number of vertices is decomposed as the latter. We have that, the old payoff of player j is $\frac{b_{j-1}}{c_{j-1}+1} + (1+a_{j-1}+a_j) + \frac{b_j}{c_{j+1}+1}$ and his new one is $\frac{b_{j-1}+b_j-1}{c_{j-1}+c_{j+1}+1} + \frac{1+(a_{j-1}+a_j+1)+a_{j+1}}{c_{j+1}+1} + \frac{b_j}{c_{j+1}+1+c_{j+1}}$. By definition, the old payoff is at least γ so $a_{j-1}+a_j \geq \gamma - 1 - \frac{b_{j-1}}{c_{j-1}+1} - \frac{b_j}{c_{j+1}+1}$. Additionally, by (ii), $a_{j+1} \leq \gamma - 1$. Thus the old payoff is greater than the new one. Similarly, in case the number of vertices between facilities u_{j-1} and u_j is decomposed as $2(a_{j-1}+a_j)+1$, the old payoff of j is also greater than his new one. Hence, a player in facility u_j has no incentive to move to a facility-neighbor u_{j+1}.

□